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STABILITY OF COUETTE FLOW IN THE CASE OF A WIDE GAP BETWEEN ROTATING CYLINDERS

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We consider the stability of the Couette flow between two rotating cylinders in the limiting case when the radius of the inner cylinder r_1 tends to zero, and its angular velocity Ω_1 increases to infinity in such a manner that $\Omega_1 r_1^2 = k_1 = \text{const}$.

The dependence of the critical Reynolds number R_* on the wave number α is represented by a neutral curve. The Couette flow loses its stability when the Reynolds number becomes supercritical and $\alpha = 3$. The eigenvector of the linearized problem is computed and used to construct an approximate Taylor vortex.

1. Statement of the problem. A viscous incompressible fluid of unit density and coefficient of viscosity ν fills the space between two concentric cylinders of radii r_1 and r_2 rotating with angular velocities Ω_1 and Ω_2. Letting r_1 tend to zero and Ω_1 to infinity in such a manner that $\Omega_1 r_1^2 = k_1$, we arrive at the limiting flow created by a vortex line of intensity k_1 distributed along the axis of the cylinder whose radius is r_2. Below we study the stablity of this flow.

In Sect. 2 we show that the problem will indeed reach its limiting value when $r_1 \to 0$.

We shall require that there is no loss of fluid across the transverse section. Then the exact solution v_0 of the Navier-Stokes equations satisfying the no-slip conditions at the boundaries represents a Couette flow

$$v_{0r} = v_{0z} = 0, \qquad v_{0\theta} = ar + 1/r, \qquad a = k_2/k_1 - 1, \qquad k_2 = \Omega_2 r_2^2 \qquad (1.1)$$

where r, θ and z denote the cylindrical coordinated.

We shall investigate the stability of the flow (1.1) towards rotationally symmetric perturbations $2\pi/\alpha$-periodic in z. Let us represent the perturbed flow by

$$\mathbf{v}'(r, z, t) = \mathbf{v}_0(r) + e^{\sigma t}\,\mathbf{v}(r, z) \qquad (1.2)$$

Inserting (1.2) into the Navier-Stokes equations and neglecting the quadratic terms, we obtain the familiar first approximation equations for small perturbations

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}=0,\quad \sigma v_r-\frac{2v_{0\theta}}{r}v_\theta=-\frac{\partial p}{\partial r}+\frac{1}{R}\left(\Delta v_r-\frac{v_r}{r^2}\right)$$

$$\sigma v_\theta+\left(\frac{dv_{0\theta}}{dr}+\frac{v_{0\theta}}{r}\right)v_r=\frac{1}{R}\left(\Delta v_\theta-\frac{v_\theta}{r^2}\right),\quad \sigma v_z=-\frac{\partial p}{\partial z}+\frac{1}{R}\Delta v_z$$

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\qquad R=\Omega_1 r_1^2/\nu \tag{1.3}$$

The functions v_r, v_θ and v_z must be $2\pi/\alpha$-periodic in z and vanish when $r=1$. We should also have $v_r=v_\theta=0$ and $v_z<\infty$ when $r=0$.

Let us assume that $a<0$ and that the "principle of alteration of stability" holds. We note that the latter principle has not, so far, been proved, although experiments corroborate it. As we know, under these assumptions the lowest value of the Reynolds number R for which the solution of the boundary value problem (1.3) with $\sigma=0$ is nontrivial, represents the critical Reynolds number R_*

We seek the solution of (1.3) with $\sigma=0$ in the form

$$\mathbf{v}(r,z)=\mathbf{v}_1(r)e^{i\alpha z} \tag{1.4}$$

Inserting (1.4) into (1.3) and eliminating the pressure as well as the perturbation of the axial velocity, we obtain the following ordinary differential equations :

$$(L-\alpha^2)^2 v_{1r}=2\alpha^2 R\,\omega(r)\,v_{1\theta},\quad (L-\alpha^2)\,v_{1\theta}=2aRv_{1r} \tag{1.5}$$

$$L=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2},\qquad \omega(r)=a+\frac{1}{r^2} \tag{1.6}$$

together with the boundary conditions

$$v_{1r}=v_{1\theta}=0\qquad \text{at}\quad r=0,\ 1$$

$$\frac{dv_{1r}}{dr}<\infty\quad \text{at}\quad r=0,\qquad \frac{dv_{1r}}{dr}=0\quad \text{at}\quad r=1 \tag{1.7}$$

The critical Reynolds number R_* is the smallest positive eigenvalue of the problem (1.5)–(1.7). We now reduce this problem to an integral equation.

Let us denote by $G^0_{1,\alpha}(r,\rho)$ the Green's function for the differential operator $(L-\alpha^2)$ with the boundary conditions $u(0)=u(1)=0$ and by $G^0_{2,\alpha}(r,\rho)$ the Green's function for the operator $(L-\alpha^2)^2$ with the boundary conditions

$$u(0)=u(1)=u'(1)=0,\qquad u'(0)<\infty$$

From the relations (1.5)–(1.7) we have

$$v_{1r}=2\alpha^2 R\int_0^1 G^0_{2,\alpha}(r,\rho)\,\omega(\rho)\,v_{1\theta}(\rho)\,d\rho \tag{1.8}$$

$$v_{1\theta}=2aR\int_0^1 G^0_{1,\alpha}(r,\rho)\,v_{1r}(\rho)\,d\rho \tag{1.9}$$

Elimination of $v_{1\theta}$ yields

$$v_{1r}=\lambda\int_0^1 G^0_{3,\alpha}(r,\rho)\,v_{1r}(\rho)\,d\rho\equiv A(v_{1r}) \tag{1.10}$$

$$\lambda = 4\alpha^2 R^2, \quad G^0_{3,\alpha}(r, \rho) = a \int_0^1 G^0_{2,\alpha}(r, s)\, G^0_{1,\alpha}(s, \rho)\, \omega(s)\, ds \qquad \text{(cont.)}$$

Thus we reduced the problem of determining R to obtaining the eigenvalues of the integral equation (1.10).

2. Passage to the limit as $r_1 \to 0$. We shall show that the number R_* represents the limiting value of the critical Reynolds numbers $R_*(r_1)$ as $r_1 \to 0$. When the gap between the cylinders is finite ($r_1 > 0$) we obtain, in a similar manner, the following equations for $R_*(r_1)$:

$$v_{1r} = 4\alpha^2 R^2(r_1) \int_0^1 G_{3,\alpha}(r, \rho)\, v_{1r}(\rho)\, \rho\, d\rho = A_{r_1}(v_{1r}) \qquad (2.1)$$

$$G_{3,\alpha}(r, \rho) = a \int_0^1 G_{2,\alpha}(r, s)\, G_{1,\alpha}(s, \rho)\, \omega(s)\, s\, ds$$

Utilizing the expression for $G_{1,\alpha}$ and $G_{2,\alpha}$ given in [1] we can show, that for any prescribed $\delta > 0$ and a sufficiently small r_1

$$|sG_{1,\alpha}(r, s) - G^0_{1,\alpha}(r, s)| < \delta, \quad |sG_{2,\alpha}(r, s) - G^0_{2,\alpha}(r, s)| < \delta$$

Therefore as $r_1 \to 0$, we have

$$\max_{0 \leqslant r, \rho \leqslant 1} |\rho G_{3,\alpha}(r, \rho) - G^0_{3,\alpha}(r, \rho)| \to 0$$

which means that the operator A_{r_1} tends to the operator A defined by (1.10) in the sense that

$$\| A_{r_1} - A \|_{C \to C} \to 0 \quad \text{when } r_1 \to 0$$

We therefore conclude that the eigenvalues of (1.10) are obtainable by a limiting process from the eigenvalues of Eq. (2.1)([2] Sect. 78).

3. The Green's functions $G^0_{1,\alpha}(r, \rho)$ and $G^0_{2,\alpha}(r, \rho)$. Let us consider the differential operator

$$(L - \alpha^2)^2 u = f \qquad (3.1)$$

with boundary conditions

$$u(0) = u(1) = u'(1) = 0, \qquad u'(0) < \infty \qquad (3.2)$$

We shall write this operator in the form [3]

$$(L - \alpha^2)^2 u = \frac{\rho_0}{r} \frac{d}{dr} \rho_1 \frac{d}{dr} \frac{\rho_2 \rho_0}{r} \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 u \qquad (3.3)$$

$$\rho_0 = \rho_2 (I_1(\alpha r))^{-1}, \qquad \rho_1 = r I^2{}_1(\alpha r)$$

Lemma 3.1. The Green's function $G^0_{2,\alpha}(r, s)$ for the boundary value problem (3.1) and (3.2) is oscillatory.

Proof of this fact is based on the results of ([4], ch. 3) and the validity of the following lemma.

Lemma 3.2. The solution of problem (3.1), (3.2) has no more sign changes in the interval (0, 1) than does the function $f(r)$.

Let us assume the opposite. Let $f(r)$ change sign n times and $u(r)$, $(n + 1)$ times in the interval (0, 1). Since conditions (3.2) hold, the function $u(r)$ has $(n + 3)$ zeros on the segment [0, 1]. By the generalized Rolle's theorem, the function

$$u_1(r) = \rho_1 \frac{d}{dr} \rho_2 u$$

has $(n+2)$ zeros on $(0, 1)$. But conditions (3.2) and $\rho_1(0) = 0$ imply that

$$u_1(1) = u_1(0) = 0$$

Therefore $u_1(r)$ has $(n+4)$ zeros on $[0, 1]$. Using now the generalized Rolle's theorem we find that the function

$$u_2(r) = \frac{\rho_0}{r} \frac{d}{dr} \rho_1 \frac{d}{dr} \frac{\rho_2 \rho_0}{r} \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 u$$

changes sign $(n+1)$ times on $(0, 1)$ which leads to a contradiction since $u_2(r) \equiv f(r)$.

Let us now write down the function

$$-G^0_{1,\alpha}(r, s) = \begin{cases} s\,[I_1(\alpha r) / I_1(\alpha)]\,[K_1(\alpha s) I_1(\alpha) - K_1(\alpha) I_1(\alpha s)] & (r \leqslant s) \\ s\,[I_1(\alpha s) / I_1(\alpha)]\,[K_1(\alpha r) I_1(\alpha) - K_1(\alpha) I_1(\alpha r)] & (r \geqslant s) \end{cases}$$

which was shown in [3] to be oscillatory. The Green's function $G^0_{2,\alpha}(r, s)$ has the form

$$G^0_{2,\alpha}(r, s) = \begin{cases} [I_1(\alpha r)\psi_1(s) - K_1(\alpha r)\psi_2(s) - \Lambda_2^{-1}(0)\psi_2(r)\psi_2(s)]\,s & (r \leqslant s) \\ [I_1(\alpha s)\psi_1(r) - K_1(\alpha s)\psi_2(r) - \Lambda_2^{-1}(0)\psi_2(r)\psi_2(s)]\,s & (r \geqslant s) \end{cases}$$

where

$$\psi_1(s) = -\Lambda_1(s) K_1(\alpha s) + \Lambda_3(s) I_1(\alpha s)$$

$$\psi_2(s) = \Lambda_1(s) I_1(\alpha s) - \Lambda_2(s) K_1(\alpha s)$$

$$\Lambda_1(s) = \int_s^1 K_1(\alpha r) I_1(\alpha r)\, r\, dr, \quad \Lambda_2(s) = \int_s^1 I_1^2(\alpha r)\, r\, dr, \quad \Lambda_3(s) = \int_s^1 K_1^2(\alpha r)\, r\, dr$$

When $\omega(r) > 0$ $(0 < r \leqslant 1)$ and $a < 0$, the kernel $G^0_{3,\alpha}(r, \rho)$, being a combination of oscillatroy kernels [4], is itself oscillatory. This means that in accordance with the results of [4] there exists a sequence of simple positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$$

for the operator $A(v_{1r})$ which is defined by (1.10).

4. Numerical results. The eigenvalues of (1.10) can be obtained using the following scheme of successive approximations

$$\lambda_{(n-1)} = \left[\int_0^1 \int_0^1 G^0_{3,\alpha}(r, \rho)\, v_{1r(n)}(\rho)\, dr d\rho\right]^{-1} \qquad (4.1)$$

$$v_{1r(n)} = \lambda_{(n-1)} \int_0^1 G^0_{3,\alpha}(r, \rho)\, v_{1r(n-1)}(\rho)\, d\rho \qquad (4.2)$$

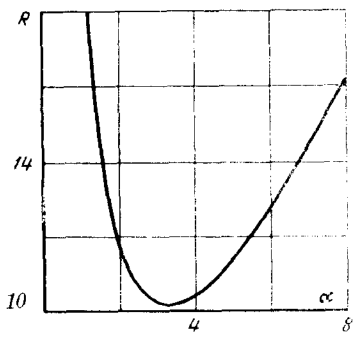

Fig. 1

Since the kernel $G^0_{3,\alpha}(r, \rho)$ is oscillatory and satisfies the positive-eigenvalue theorem [5], sequences (4.1) and (4.2) converge to the smallest eigenvalue of (1.10) and to the corresponding eigenfunction, respectively.

Computing the critical number R_* for various α we obtain, in the case when the outer cylinder is at rest $(\Omega_2 = 0)$, the neutral curve shown in Fig. 1. Computations were performed on the "Minsk-12" digital computer up to three or four significant figures in the value of R_*. The value of the wavenumber α_* corresponding to $\min_\alpha R_*(\alpha)$ falls between $\alpha = 3$ and $\alpha = 3.5$.

The method given in [1] was used to calculate the first eigenvalue in the spectrum of stability of the Couette flow for $\alpha = 3$ at the supercritical Reynolds numbers $R = R_* + \varepsilon^2$, and it was found to be

$$\sigma = 2.21\,\varepsilon^2 + O(\varepsilon^4)$$

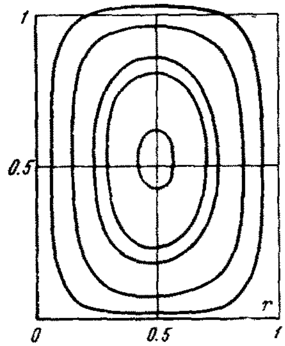

Fig. 2

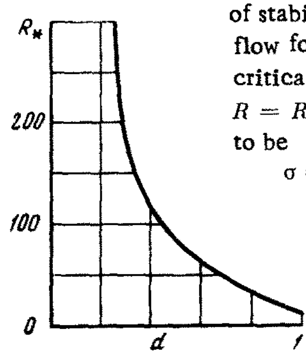

Fig. 3

It follows that when R becomes supercritical, the Couette flow loses its stability. When $\sigma = 0$, the eigenvector of linearized problem (1.3) and the first term of the parametric expansion of the secondary flow, i.e. the Taylor vortex which appears when the Couette flow loses stability, coincide to within the constant multiplier. We computed this eigenvector in order to construct the approximate Taylor vortex shown on Fig. 2. Its v_{1r} component and R_* were obtained from the scheme (4.1) and (4.2), its $v_{1\theta}$ component was found from (1.9) and for v_{1z} we obtain

$$v_{1z} = -2R_*\alpha \int_0^1 \frac{1}{r}\left(G^0_{2,\alpha}(r,\rho) + r\,\frac{\partial G^0_{2,\alpha}(r,\rho)}{\partial r}\right)\omega(\rho)\,v_{1\theta}(\rho)\,d\rho$$

Table 1 gives the results obtained. Fig. 3 depicts the dependence of the critical numbers $R_*(\alpha_*) = \min_\alpha R_*(\alpha)$ (with $\alpha \approx 3$) on the quantity $d = (r_2 - r_1)/r_2$. For $d = 1/2$ and $d = 1/3$ the values for $R_*(\alpha_*)$ were taken from [1 and 6], for $d \approx 0.12$ from the data obtained by G. Taylor (quoted in e.g. [7]) and for $d = 1$, from the present paper.

Table 1

r	v_{1r}	$v_{1\theta}$	v_{1z}
0.0625	1.6463	0.8760	−16.910
0.1250	2.9759	1.6768	−13.973
0.1875	3.9019	2.3460	−10.835
0.2500	4.4435	2.8495	−7.8635
0.3125	4.6421	3.1725	−5.1998
0.3750	4.5553	3.3167	−2.9429
0.4375	4.2472	3.2962	−1.1169
0.5000	3.7800	3.1336	0.2927
0.5625	3.2103	2.8566	1.3181
0.6250	2.5884	2.4953	1.9971
0.6875	1.9583	2.0796	2.3646
0.7500	1.3590	1.6374	2.4475
0.8125	0.8268	1.1931	2.2607
0.8750	0.3971	0.7656	1.8047
0.9375	0.1073	0.3670	1.0629

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THE EULER APPROXIMATION FOR COLLISIONLESS POLYDISPERSE SUSPENSIONS

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The kinetic equations for various fractions of the dispersed phase of a polydisperse suspension and the system of dynamic equations defining the motion of the suspension as a set of interpenetrating continua are formulated. It is assumed that the suspension is "collisionless", i. e. that its particles interact largely by way of the random velocity and pressure fields in the dispersion medium. The relations characterizing the structure of the random pulsations of the suspension phases ("pseudoturbulence") are considered without allowance for the derivatives of the dynamic variables describing the mean motion. This makes it possible to obtain the dynamic equations in an approximation analogous to the Euler approximation in the hydromechanics of single-phase media. The equations of pseudo-turbulent particle energy transfer which close the system of dynamic equations are written out in the same approximation.

A hydrodynamic model of a polydisperse suspension which adequately describes its mechanical behavior in the continuum approximation can be constructed by a natural generalization of the method already applied to a monodisperse suspension (e. g. see [1]). To avoid repetition, many of the concepts discussed in detail in the case of a monodisperse suspension are used here without further explanation. For clarity we begin with the case where the disperse phase can be represented as a collection of a finite set of fractions. The results thus obtained are then extended to suspensions with continuous